TD0 - Séries

Exercice 1

1/ $u_n = \sqrt{n+1} - \sqrt{n} = \dfrac{(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{(\sqrt{n+1} + \sqrt{n})}$

$= \dfrac{n+1-n}{\sqrt{n+1} + \sqrt{n}} = \dfrac{1}{\sqrt{n}\left(1 + \sqrt{1 + 1/n}\right)} \xrightarrow[n \to +\infty]{} 0$

2/ $u_n = \left(1 + \dfrac{1}{n}\right)^n = e^{n \ln\left(1 + \frac{1}{n}\right)}$

$= e^{n\left(\frac{1}{n} + o\left(\frac{1}{n}\right)\right)} = e^{1 + o(1)} \xrightarrow[n \to +\infty]{} e$

3/ $u_n = \sqrt{n}\, e^{-\sqrt{\ln(n)}}$

$= e^{\ln(\sqrt{n})} \cdot e^{-\sqrt{\ln(n)}}$

$= e^{\frac{1}{2}\ln(n) - \sqrt{\ln(n)}}$

$= e^{\ln(n)\left(\frac{1}{2} - \frac{1}{\sqrt{\ln(n)}}\right)} \xrightarrow[n \to +\infty]{} +\infty$

(où $\frac{1}{2} - \frac{1}{\sqrt{\ln(n)}} \xrightarrow[n \to +\infty]{} \frac{1}{2}$)

4/ $u_n = \dfrac{\sin(n^2)}{n}$

$|u_n| \leqslant \dfrac{1}{n} \xrightarrow[n \to +\infty]{} 0 \Rightarrow \lim\limits_{n \to +\infty} u_n = 0$

5/ $u_n = \dfrac{a^n - b^n}{a^n + b^n}$ $\quad (a, b) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$

1^er cas : $a < b$ $\left(\dfrac{a}{b} < 1\right)$

$$u_n = \frac{b^n\left(\left(\frac{a}{b}\right)^n - 1\right)}{b^n\left(\left(\frac{a}{b}\right)^n + 1\right)} = \frac{\left(\frac{a}{b}\right)^n - 1}{\left(\frac{a}{b}\right)^n + 1}$$

or $\left|\frac{a}{b}\right| < 1 \Rightarrow \left(\frac{a}{b}\right)^n \xrightarrow[n \to +\infty]{} 0$

d'où $\lim_{n \to +\infty} u_n = -1$

<u>2ième cas</u> $a > b$ $\left(\frac{b}{a} < 1\right)$

$$u_n = \frac{a^n\left(1 - \left(\frac{b}{a}\right)^n\right)}{a^n\left(1 + \left(\frac{b}{a}\right)^n\right)} \xrightarrow[n \to +\infty]{} 1$$

<u>3ième cas</u> $a = b$ $u_n = 0 \xrightarrow[n \to +\infty]{} 0$

Exercice 2

1] $u_n = \left(\ln\left(1 + e^{-n^2}\right)\right)^{1/n}$

$$= e^{1/n \ln\left(\ln\left(1 + e^{-n^2}\right)\right)} \qquad e^{-n^2} \xrightarrow[n \to +\infty]{} 0$$

$$= e^{\frac{1}{n} \ln\left(e^{-n^2} - \frac{\left(e^{-n^2}\right)^2}{2} + o\left(e^{-2n^2}\right)\right)}$$

$$= e^{1/n \ln\left(e^{-n^2}\left(1 - \frac{e^{-n^2}}{2} + o\left(e^{-n^2}\right)\right)\right)}$$

$$= e^{1/n\left(-n^2 + \ln\left(1 - \frac{e^{-n^2}}{2} + o\left(e^{-n^2}\right)\right)\right)}$$

$$= e^{-n + 1/n\left(-\frac{e^{-n^2}}{2} + o\left(e^{-n^2}\right)\right)}$$

$$= e^{-n} \cdot \underbrace{e^{-1/n\left(\frac{e^{-n^2}}{2} + o\left(e^{-n^2}\right)\right)}}_{\xrightarrow[n \to +\infty]{} 1} \underset{+\infty}{\sim} e^{-n}$$

2] $u_n = \left(\frac{e^{n}}{1+e^{-n}}\right)$

$$= \left(\frac{1+e^{-n}}{e^{n}}\right)^{-n}$$

$$= \left(e^{-n}+e^{-2n}\right)^{-n}$$

$$= \left(e^{-n}\left(1+e^{-n}\right)\right)^{-n}$$

$$= \left(e^{-n}\right)^{-n}\left(1+e^{-n}\right)^{-n}$$

$$= e^{n^2}\, e^{-n\ln(1+e^{-n})}$$

$$= e^{n^2}\, e^{-n\left(e^{-n}+o(e^{-n})\right)}$$

$$= e^{n^2}$$

Exercice 3

1] $f_1(x) = x^{3/2}\left(\sqrt{1+x^{-1/2}} - 1\right)$

$$= x^{3/2}\left(1+\frac{1}{2}\frac{1}{\sqrt{x}}+o\left(\frac{1}{\sqrt{x}}\right)-1\right)$$

$$= \frac{1}{2}x + o(x)$$

$$\underset{+\infty}{\sim} \frac{1}{2}x.$$

2] $$\frac{\sqrt{x^4+1}-\sqrt{x^4-1}}{\sqrt{x^2+1}-\sqrt{x^2-1}} = \frac{\sqrt{x^2+1}+\sqrt{x^2-1}}{\sqrt{x^4+1}+\sqrt{x^4-1}}$$

$$= \frac{x\left(\sqrt{1+1/x^2}+\sqrt{1-1/x^2}\right)}{x^2\left(\sqrt{1+1/x^4}+\sqrt{1-1/x^4}\right)} \underset{+\infty}{\sim} 1/x.$$

3] $$f_3(x) = \frac{e^{1/x} - \cos 1/x}{1 - \sqrt{1 - 1/x^2}}$$

$$= \frac{1 + 1/x + \frac{1}{2x^2} - 1 + 1/2x^2 + o(1/x^2)}{1 - \left(1 - \frac{1}{2x^2} + o(1/x^2)\right)}$$

$$= \frac{1/x + o(1/x^2)}{\frac{1}{2x^2} + o(1/x^2)}$$

$$= \frac{2x + o(1)}{1 + o(1)} \underset{+\infty}{\sim} 2x.$$

4] $$f_4(x) = \frac{1}{\sqrt{x}} - \sqrt{x}\,\sin(1/x)$$

$$= \frac{1}{\sqrt{x}} - \sqrt{x}\left(\frac{1}{x} - \frac{1}{6x^3} + o(1/x^3)\right)$$

$$= 1/\sqrt{x} - 1/\sqrt{x} + \frac{1}{6x^{5/2}} + o\left(\frac{1}{x^{5/2}}\right)$$

$$\underset{+\infty}{\sim} \frac{1}{6x^{5/2}}$$

5] $$f_5(x) = e^{1/x} - e^{\frac{1}{x+1}}$$

$$= e^{1/x} - e^{\frac{1}{x(1+1/x)}}$$

$$f_5(x) = e^{\frac{1}{x}} - e^{\frac{1}{x+1}}$$

$$= e^{\frac{1}{x}} - e^{\frac{1}{x(1+1/x)}}$$

$$= e^{\frac{1}{x}} - e^{\frac{1}{x}\frac{1}{1+1/x}}$$

$$= e^{\frac{1}{x}} - e^{\frac{1}{x}\left(1 - \frac{1}{x} + o\left(\frac{1}{x}\right)\right)}$$

$$= e^{\frac{1}{x}} - e^{\frac{1}{x} - \frac{1}{x^2} + o\left(\frac{1}{x^2}\right)}$$

$$= e^{\frac{1}{x}}\left(1 - e^{-\frac{1}{x^2} + o\left(\frac{1}{x^2}\right)}\right)$$

$$= e^{\frac{1}{x}}\left(1 - \left(1 - \frac{1}{x^2} + o\left(\frac{1}{x^2}\right)\right)\right)$$

$$= e^{\frac{1}{x}}\left(\frac{1}{x^2}(1 + o(1))\right)$$

$$= \frac{1}{x^2} e^{\frac{1}{x}}(1 + o(1)) \underset{+\infty}{\sim} \frac{e^{\frac{1}{x}}}{x^2}$$

Rq. $\frac{e^{1/x}}{x^2} = \frac{1}{x^2}\left(1 - 1/x + o(1/x)\right)$

$$= \frac{1}{x^2} - \frac{1}{x^3} + o\left(\frac{1}{x^3}\right)$$

$$\underset{+\infty}{\sim} \frac{1}{x^2}.$$

Exercice 5

1] $\lim_{x \to 0} f(x)$ avec $f(x) = \frac{4\sin^3 x + x - 4(\cos x - 1)}{3x^2 + e^x - 1}$

DL de f à l'ordre 3 en 0.

$$f(x) = \frac{4\left(x - x^3/6 + o(x^3)\right)^3 + x - 4\left(1 - \frac{x^2}{2} + o(x^3) - 1\right.}{3x^2 + 1 + x + x^2/2 + x^3/6 + o(x^3)}$$

$$f(x) = \frac{4x^3 + x + 4\,x^2/2 + o(x^3)}{3x^2 + x + x^2/2 + x^3/6 + o(x^3)}$$

$$= \frac{1 + 2x + 4x^2 + o(x^2)}{1 + \frac{7}{2}x + 1/6\, x^2 + o(x^2)} \xrightarrow[x \to 0]{} 1.$$

2]

$$g(x) = \frac{(1 - e^x)\sin x}{e^x - \operatorname{ch} x}$$

DL de g à l'ordre 2 en 0.

$$g(x) = \frac{(1 - 1 - x - x^2/2 + o(x^2))(x + o(x^2))}{1 + x + x^2/2 - 1 - x^2/2 + o(x^2)}$$

$$= \frac{-x^2 + o(x^2)}{x + o(x^2)} = \frac{-x + o(x)}{1 + o(x)} \xrightarrow[x \to 0]{} [illegible]$$

Exercice 5

3/ $\lim_{x \to 1} \dfrac{x^x - 1}{1 - x + \ln(1+x)} = \dfrac{1 - 1}{1 - 1 + \ln(2)} =$

4/ $\lim_{x \to +\infty} \sqrt{x + \sqrt{x}} - \sqrt{x} = \lim_{x \to +\infty} \dfrac{x + \sqrt{x} - x}{\sqrt{x+\sqrt{x}} + \sqrt{x}}$

$= \lim_{x \to +\infty} \dfrac{\sqrt{x}}{\sqrt{x + \sqrt{x}} + \sqrt{x}} = \lim_{x \to +\infty} \dfrac{1}{\sqrt{1 + \frac{1}{\sqrt{x}}} + 1}$

$= \dfrac{1}{2}$

5/ $\lim_{x \to +\infty} \dfrac{(x^x)^x}{x^{(x^x)}}$

$= \lim_{x \to +\infty} \dfrac{e^{x \ln(x^x)}}{e^{x^x \ln(x)}} = \lim_{x \to +\infty} \dfrac{e^{x^2 \ln(x}}{e^{x^x \ln}}$

$= \lim_{x \to +\infty} e^{(x^2 - x^x)\ln(x)}$

$= \lim_{x \to +\infty} e^{x^2 (1 - e^{(x-2)\ln x})\ln(x)}$

$= 0$

6/ $\lim_{x \to +\infty} \left(\cos\left(\frac{1}{x}\right)\right)^{x^2} = \lim_{x \to +\infty} e^{x^2 \ln(\cos 1/x}$

$= \lim_{x \to +\infty} e^{x^2 \ln\left(1 - \frac{1}{2x^2} + o\left(\frac{1}{x^2}\right)\right)}$

$= \lim_{x \to +\infty} e^{-\frac{1}{2} + o(1)} = e^{-\frac{1}{2}}$

7/ $\lim_{x \to 1} \frac{(2x - x^3)^{1/3} - x^{1/2}}{1 - x^{3/4}}$

on pose $y = 1 - x$ $\quad x = 1 - y$

$(2x - x^3)^{1/3} = (2(1-y) - (1-y)^3)^{1/3}$

$= (1 + \boxed{y - 3y^2 + y^3})^{1/3}$

$\parallel$ Y $\quad Y \to 0$

$(1 + Y)^{1/3} = 1 + \frac{1}{3} Y - \frac{1}{9} Y^2 + \frac{5}{243} Y^3 + o(Y^3)$

$(1 + y - 3y^2 + y^3)^{1/3}$

$= 1 + \frac{1}{3} y - y^2 + \frac{1}{3} y^3 - \frac{1}{9} y^2 + \frac{2}{3} y^3 + \frac{5}{243} y^3 + o(y^3)$

$= 1 + \frac{1}{3} y - \frac{10}{9} y^2 + (1 + \frac{5}{243}) y^3 + o(y^3)$

$x^{\frac{1}{2}} = (1 - y)^{1/2} = 1 - \frac{1}{2} y - \frac{1}{8} y^2 - \frac{1}{16} y^3 + o(y^3)$

$x^{3/4} = (1 - y)^{3/4} = 1 - \frac{3}{4} y - \frac{3}{32} y^2 - \frac{5}{344} y^3 + o(y^3)$

$\lim_{x \to 1} \frac{(2x - x^3)^{1/3} - x^{1/2}}{1 - x^{3/4}}$

$= \lim_{y \to 0} \frac{(1 + y - 3y^2 + y^3)^{1/3} - (1 - y)^{1/2}}{1 - (1 - y)^{3/4}}$

$= \lim_{y \to 0} \frac{(1/3 + 1/2) y + o(y)}{3/4\, y + o(y)} = \lim_{y \to 0} \frac{\frac{5}{6} y + o(y)}{\frac{3}{4} y + o(y)}$

$= \lim_{y \to 0} \frac{5/6 + o(1)}{3/4 + o(1)} = \frac{5}{6} \times \frac{4}{3} = \frac{10}{9}$

8/ $\lim_{x\to 0} \left(\frac{x}{\sin x}\right)^{\frac{\sin x}{x-\sin x}}$

$= \lim_{x\to 0} \exp\left(\frac{\sin x}{x-\sin x} \ln\left(\left(\frac{\sin x}{x}\right)^{-1}\right)\right)$

$= \lim_{x\to 0} \exp\left(-\frac{x - x^3/6 + o(x^3)}{x - x + x^3/6 + o(x^3)} \ln\left(\frac{x - x^3/6 + o(x^3)}{x}\right)\right)$

$= \lim_{x\to 0} \exp\left(-\frac{1 - x^2/6 + o(x^2)}{x^2/6 + o(x^2)} \ln\left(1 - x^2/6 + o(x^2)\right)\right)$

$= \lim_{x\to 0} \exp\left(-\frac{1 - x^2/6 + o(x^2)}{x^2/6 + o(x^2)} \cdot \left(-x^2/6 + o(x^2)\right)\right)$

$= \lim_{x\to 0} \exp\left(-\frac{1 - x^2/6 + o(x^2)}{1 + o(1)} \left(-1 + o(1)\right)\right)$

$= e$

9/ $\lim_{x\to 0} \left(\frac{a^x + b^x}{2}\right)^{1/x} = \lim_{x\to 0} e^{\frac{1}{x}\ln\left(\frac{a^x+b^x}{2}\right)}$

$= \lim_{x\to 0} e^{\frac{1}{x}\left(\ln\left(\frac{a^x+b^x}{2}\right)\right)}$

$X \to 0$, $X = x\ln(a)$; $Y \to 0$, $Y = x\ln(b)$

$= \lim_{x\to 0} e^{\frac{1}{x}\ln\left(\frac{1}{2}\left(e^{x\ln(a)} + e^{x\ln(b)}\right)\right)}$

$= \lim_{x\to 0} e^{\frac{1}{x}\ln\left(\frac{1}{2}\left(1 + x\ln(a) + 1 + x\ln(b) + o(x)\right)\right)}$

$= \lim_{x\to 0} e^{1/x \ln\left(1 + x\frac{\ln(a)+\ln(b)}{2} + o(x)\right)}$

$Z = x\frac{\ln(a)+\ln(b)}{2} + o(x)$, $Z \to 0$

$= \lim_{x\to 0} e^{\frac{1}{x}\cdot\left(x\frac{\ln(a)+\ln(b)}{2} + o(x)\right)}$

$= e^{\frac{\ln(a)+\ln(b)}{2}} = e^{\ln(\sqrt{ab})} = \sqrt{ab}$

10/ $\lim_{x \to 0} \dfrac{(1-\cos x)(1+2x)}{x^2 - x^4}$

$= \lim_{x \to 0} \dfrac{\left(x^2/2 + o(x^2)\right)(1+2x)}{x^2 - x^4}$

$= \lim_{x \to 0} \dfrac{\left(1/2 + o(1)\right)(1+2x)}{1 - x^2} = 1/2$

11/ $\ln\left(1 - \dfrac{\sqrt{x+1}}{x+2}\right) \underset{+\infty}{\sim} -\dfrac{\sqrt{x+1}}{x+2}$

$\sqrt{4x+1} \cdot \ln\left(1 - \dfrac{\sqrt{x+1}}{x+2}\right) \underset{+\infty}{\sim} \dfrac{\sqrt{4x+1}\sqrt{x+1}}{x+2}$

$x \to +\infty$

$\sqrt{4} = 2$

d'où $\lim_{x \to +\infty} \sqrt{4x+1} \cdot \ln\left(1 - \dfrac{\sqrt{x+1}}{x+2}\right) = 2.$

12/ $x(3+x)\dfrac{\sqrt{3+x}}{\sqrt{x}\sin\sqrt{x}} \underset{0}{\sim} x(3) \cdot \dfrac{\sqrt{3}}{\sqrt{x}\sqrt{x}} = 3\sqrt{3}$

d'où $\lim_{x \to 0} x(3+x)\dfrac{\sqrt{3+x}}{\sqrt{x}\sin\sqrt{x}} = 3\sqrt{3}$

13/ $\lim_{x \to 0} \exp(1/x^2) - \exp\left(\left(\dfrac{1}{x+1}\right)^2\right)$

$e^{1/x^2} - e^{1/(x+1)^2} = e^{1/x^2}\left(1 - e^{-1/x^2 + \frac{1}{(x+1)^2}}\right)$

$= e^{1/x^2}\left(1 - e^{\frac{-2x-1}{x^2(x+1)^2}}\right)$; $x \to 0$: $e^{1/x^2} \to +\infty$; $1 - e^{\frac{-2x-1}{x^2(x+1)^2}} \to 1$

$=$

Exercice 4

$$f(x) = \left[\frac{\ln(1+x)}{\ln(x)}\right]^{x \ln(x)}$$

1/ $$f(x) = e^{x \ln x \ln\left(\frac{\ln(1+x)}{\ln(x)}\right)}$$

$$\ln(f(x)) = x \ln x \ln\left(\frac{\ln(x(1+1/x))}{\ln x}\right)$$

$$= x \ln(x) \ln\left(\frac{\ln(x) + \ln(1+1/x)}{\ln(x)}\right)$$

$$= x \ln(x) . \ln\left(1 + \frac{1}{\ln(x)} \ln(1+1/x)\right)$$

$$= x \ln(x) . \ln\left(1 + \frac{1}{\ln(x)}\left(\frac{1}{x} + o(1/x)\right)\right)$$

$$= x \ln(x) \ln\left(1 + \frac{1}{x \ln x} + o\left(\frac{1}{x \ln x}\right)\right)$$

$Y \to 0$

$$= x \ln(x) . \left(\frac{1}{x \ln(x)} + o\left(\frac{1}{x \ln(x)}\right)\right)$$

$$= 1 + o(1)$$

$$\ln(f(x)) \underset{+\infty}{\sim} 1$$

$$\lim_{x \to +\infty} \ln(f(x)) = 1 \in \mathbb{R}$$

en appliquant la fct exponentielle on obtien

$$\lim_{x \to +\infty} f(x) = e.$$

Exercice 6

1] $\cos x - 1 \underset{0}{\sim} -\frac{x^2}{2}$

$\sin(x) \underset{0}{\sim} x$.

$\frac{\cos x - 1}{\sin x} \underset{0}{\sim} -\frac{x}{2} \xrightarrow[x \to 0]{} 0$

$\Rightarrow \lim_{x \to 0} \frac{\cos x - 1}{\sin x} = 0 \Rightarrow \cos x - 1 = o(\sin x$

2] $\ln(1+2x) \underset{0}{\sim} 2x$.

$\frac{\ln(1+2x)}{x \ln x} \underset{0}{\sim} \frac{2x}{x \ln(x)} \xrightarrow[x \to 0]{} 0$

donc $\ln(1+2x) = o(x \ln(x))$

3] $x^{\ln(x)} = e^{\ln(x).\ln(x)} = e^{(\ln(x))^2}$

$\ln(x)^x = e^{x.\ln(\ln(x))}$

on pose $X = \ln(x)$, $x = e^X$

$x^{\ln(x)} = e^{X^2}$

$\ln(x)^x = e^{e^X \ln(X)}$

$\frac{e^{X^2}}{e^{e^X \ln(X)}} = e^{X^2 - e^X \ln(X)}$ $\xrightarrow[X \to +\infty]{} +\infty$

$= e^{X^2\left(1 - \frac{e^X \ln(X)}{X^2}\right)}$ $x \ln(x) = o$

$\Rightarrow x^{\ln x} = o((\ln(x))^x)$. $\xrightarrow[X \to +\infty]{} 0$ //

4]. $\left| \frac{\sqrt{x^2+3x}\, \ln(x^2) \sin x}{x \ln(x)} \right| \leq \frac{x\sqrt{1+3/x}\,(2\ln x)}{x \ln x}$

$= 2\sqrt{1+3/x} \leq$ (with $\sqrt{1+3/x} \leq 2$)

donc $\sqrt{x^2+3x}\, \ln(x^2) \sin x = O(x \ln x)$

Exercice 7

$f, g : \mathbb{R} \longrightarrow \mathbb{R}$. $\lim_{x\to+\infty} f(x) = +\infty$

1/ On suppose que $g \underset{+\infty}{=} o(f)$.

Mq $\exp(g) \underset{+\infty}{=} o(\exp(f))$.

On a : $g \underset{+\infty}{=} o(f) \iff \exists$ une fct $\varepsilon : \mathbb{R} \to \mathbb{R}$

tq : $\begin{cases} g(x) = \varepsilon(x) f(x) \\ \lim_{x\to+\infty} \varepsilon(x) = 0 \end{cases}$

$$\frac{\exp(g(x))}{\exp(f(x))} = \frac{\exp(\varepsilon(x) f(x))}{\exp(f(x))}$$

$$= \exp\left(f(x) \left(\varepsilon(x) - 1 \right) \right)$$

(hypothèse : $f(x) \xrightarrow[x\to+\infty]{} +\infty$; $\varepsilon(x) - 1 \xrightarrow[x\to+\infty]{} -1$)

donc $\lim_{x\to+\infty} \frac{\exp(g(x))}{\exp(f(x))} = 0 \Rightarrow \exp(g) = o(\exp(f))$

2/ Montrons que

si $\exp(g) \underset{+\infty}{=} o(f) \not\Rightarrow g \underset{+\infty}{=} o(f)$

soit $g(x) = x^2 - x$

$f(x) = x^2$

on a $\frac{e^{g(x)}}{e^{f(x)}} = \frac{e^{x^2 - x}}{e^{x^2}} = e^{-x} \xrightarrow[x \to +\infty]{} 0$

donc $\exp(g) = o(\exp(g))$

mais $f(x) \underset{+\infty}{\sim} g(x)$ et $g \underset{+\infty}{\neq} o(f)$.